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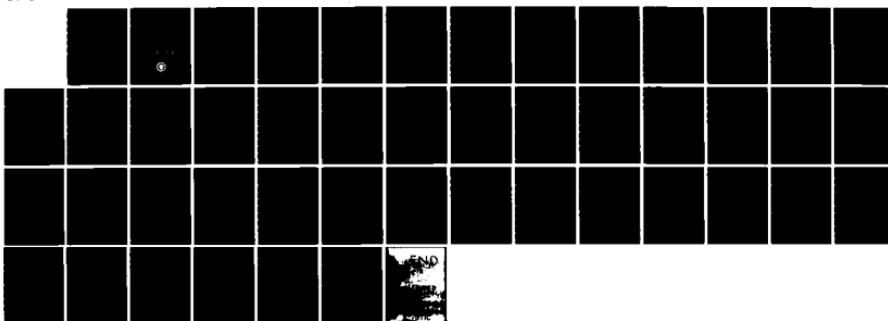
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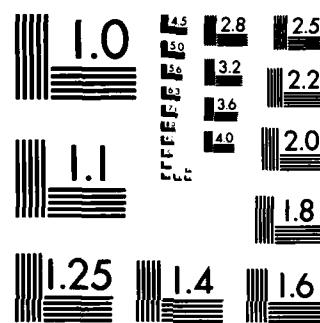
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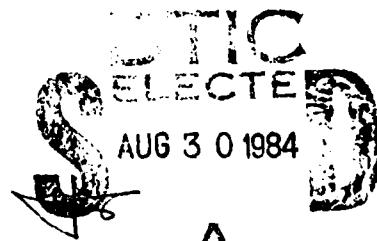
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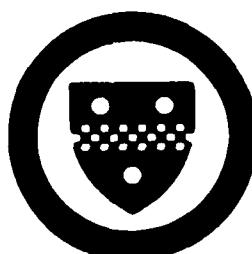
SOME WEAK AND STRONG LAWS OF LARGE
NUMBERS FOR $D[0,1]$ - VALUED RANDOM
VARIABLES

Xiang Chen Wang and M. Bhaskara Rao*



Center for Multivariate Analysis

University of Pittsburgh



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Xiang Chen Wang and M. Bhaskara Rao*

July 1984

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SOME WEAK AND STRONG LAWS OF LARGE NUMBERS FOR $D[0,1]$ -VALUED
RANDOM VARIABLES

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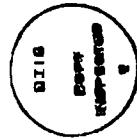
Abstract

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SOME WEAK AND STRONG LAWS OF LARGE NUMBERS FOR $D[0,1]$ -VALUED
RANDOM VARIABLES

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1. INTRODUCTION

Let $D[0,1]$ be the space of all real-valued functions x defined on the unit interval $[0,1]$ that are right continuous over $[0,1]$, possess left-hand limits at every t in $(0,1]$ and left continuous at $t = 1$. $D[0,1]$ is a linear space and is a model for quite a number of stochastic processes containing jumps. Every x in $D[0,1]$ is bounded and has atmost countably many discontinuities. $D[0,1]$ can be equipped with two topologies. One topology (norm topology) comes from the norm $\| \cdot \|$ on $D[0,1]$ defined by $\| x \| = \sup_{t \in [0,1]} |x(t)|$ for x in $D[0,1]$. The Banach space $(D[0,1], \| \cdot \|)$ is not separable. Another topology (Skorokhod topology) comes from the Skorokhod metric d on $D[0,1]$.

(For the definition of Skorokhod metric and its properties, see Billingsley [1, p.109-153].) The metric space $(D[0,1], d)$ is topologically complete and separable, but not a linear topological space. For a study of stochastic convergence in $D[0,1]$, many of the classical techniques become inapplicable when $D[0,1]$ is equipped with either of the two topologies described above. A study of topological convergence in $D[0,1]$ in either of the two topologies above would help to pave a way for a study of stochastic convergence in $D[0,1]$.

In Section 2, we examine some conditions under which for a given sequence $x_n, n \geq 0$ in $D[0,1]$, (i), (ii) and (iii) described below are equivalent.

(i) $x_n, n \geq 1$ converges to x_0 pointwise on some dense subset S of $[0,1]$, i.e., $\lim_{n \rightarrow \infty} x_n(t) = x_0(t)$ for every t in S .

(ii) $x_n, n \geq 1$ converges in the Skorokhod topology to x_0 , i.e., $\lim_{n \rightarrow \infty} d(x_n, x_0) = 0$.

(iii) $x_n, n \geq 1$ converges to x_0 in the norm topology, i.e.

$$\lim_{n \rightarrow \infty} ||x_n - x_0|| = 0.$$

One of the results in Section 2 establishes the equivalence of (i), (ii) and (iii) when the sequence $x_n, n \geq 1$ belongs to a compact convex subset (in the Skorokhod topology) of $D[0,1]$ generalizing a result of Daffer [3]. In simplistic terms, what this result means is that if $x_n, n \geq 1$ and the convex hull of the set $x_n, n \geq 1$ are relatively compact in the Skorokhod topology of $D[0,1]$, then the pointwise convergence of the sequence $x_n, n \geq 1$ implies norm convergence of the sequence $x_n, n \geq 1$.

Of primary importance, we discuss stochastic convergence in $D[0,1]$ in Section 3. Let $D[0,1]$ be equipped with the Borel σ -field B_1 generated by the Skorokhod topology. Let (Ω, \mathcal{B}, P) be a probability space. A map $X : \Omega \rightarrow D[0,1]$ is said to be a random element if $X^{-1}(B) \in \mathcal{B}$ for every B in B_1 . X is a random element if and only if $\lambda \mapsto X(\lambda)(t) : \Omega \rightarrow \mathbb{R}$, the real line

is a real random variable for every t in $[0,1]$. Consequently, if X and Y are two random elements, then $X + Y$ is also a random element. Also, if $E\|X\| < \infty$, then $EX(\cdot)(t)$ is finite for every t in $[0,1]$, and the function $x(t) = EX(\cdot)(t)$ for t in $[0,1]$ defines an element x in $D[0,1]$ and is called the expected value of X , denoted by EX . Let x_n , $n \geq 1$ be a sequence random elements defined on Ω taking values in $D[0,1]$ and a_{nk} , $1 \leq k \leq n$, $n \geq 1$ be a triangular array of real numbers. We seek conditions under which the sequence $\sum_{k=1}^n a_{nk} x_k$, $n \geq 1$ of weighted sums converges in probability (Weak Law of Large Numbers), or in the p -th mean, or a.e. $[P]$ (Strong Law of Large Numbers). Of special interest, we examine whether pointwise Weak Laws of Large Numbers would force the validity of corresponding Weak Laws of Large Numbers for the given sequence of random elements in $D[0,1]$. More specifically, we observe that we have a sequence $\sum_{k=1}^n a_{nk} x_k(\cdot)(t)$, $n \geq 1$ of real random variables and examine whether the following two statements are equivalent.

(i) Weak Law of Large Numbers holds for the sequence

$$\sum_{k=1}^n a_{nk} x_k(\cdot)(t), \quad n \geq 1$$

for every t in $[0,1]$.

(ii) Weak Law of Large Numbers holds for the sequence

$$\sum_{k=1}^n a_{nk} x_k, \quad n \geq 1$$

either in the Skorokhod topology or norm topology.

This type of study has been carried out in the literature for B -valued random variables, where B is a separable Banach space. See Taylor [8] and Wang and Bhaskara Rao [12]. Virtually, equivalence of (i) and (ii), in this case, is guaranteed if the sequence $x_n, n \geq 1$ is uniformly compactly 1st-order integrable, i.e., given $\epsilon > 0$ there exists a compact subset K of B such that

$$\int_{\{x_n \in K^c\}} ||x_n|| \, dP < \epsilon$$

for every $n \geq 1$. The methods used in this case do not carry out to the space $D[0,1]$. It has been felt that in order to achieve the equivalence of (i) and (ii) in the setting of $D[0,1]$ -space, one needs a stronger condition than uniformly compactly 1st-order

integrability. Accordingly, Taylor and Daffer [10, p.97] introduced the following condition.

A sequence $x_n, n \geq 1$ of random elements in $D[0,1]$ is said to satisfy (CT) condition if for every $\epsilon > 0$ there exists a compact convex (in the Skorokhod topology) subset K of $D[0,1]$ such that

$$\int_{\{x_n \in K^c\}} ||x_n|| \, dP < \epsilon$$

for every $n \geq 1$. Using Theorem 3.1 of this paper, one can characterize this (CT) condition. It follows that $x_n, n \geq 1$ satisfies (CT) condition if and only if $x_n, n \geq 1$ is uniformly convex tight, i.e., given $\epsilon > 0$ there exists a compact convex subset K of $D[0,1]$ such that $P\{x_n \in K^c\} < \epsilon$ for every $n \geq 1$, and $x_n, n \geq 1$ is uniformly integrable. As a consequence, for a sequence $x_n, n \geq 1$ of random elements in $D[0,1]$, it follows that (A) \iff (B) \iff (C) in the following.

(A) (a) $x_n, n \geq 1$ is uniformly convex tight.

(b) $\sup_{n \geq 1} E||x_n||^p < \infty$ for some $p > 1$.

(3) (a) $x_n, n \geq 1$ is uniformly convex tight.

(b) $x_n, n \geq 1$ is uniformly dominated by a real random variable X , i.e., $P\{||x_n|| \geq a\} \leq P\{|X| \geq a\}$ for every $a \geq 0$ and $n \geq 1$, with the additional property that $E|X| < \infty$.

(c) $x_n, n \geq 1$ satisfies (CT) condition.

The implication (A) \implies (C) was observed by Taylor and Daffer [10, p.97].

Just as compact sets play a crucial role in the form of uniform tightness in extending limit theorems on the real line to the setting of separable Banach spaces, it is the compact convex sets in $D[0,1]$ that play a crucial role in the setting of $D[0,1]$ -space. We denote by K the collection of all subsets K of $D[0,1]$ such that K and $co(K)$, the convex hull of K , are relatively compact in the Skorokhod topology.

In Theorem 3.4 in this paper, we show that the validity of Weak Law of Large Numbers pointwise is equivalent to the validity

of Weak Law of Large Numbers in the norm topology if the sequence $x_n, n \geq 1$ satisfies (CT) condition. This result generalizes Theorem 1 of Taylor and Daffer [10, p.97].

Under (CT) condition, we establish some Strong Laws of Large Numbers. Marcinkiewicz-Zygmund-Kolmogorov's Strong Law of Large Numbers and Brunk-Chung's Strong Law of Large Numbers are established in the $D[0,1]$ -space setting. In Section 3, we also obtain an analogue of Rohatgi's Strong Law of Large Numbers in the setting of $D[0,1]$ -space. See Rohatgi [7, Theorem 2, p.306].

It must be emphasized that from the Weak and Strong Laws of Large Numbers established in the setting of $D[0,1]$ -space, one can derive corresponding Weak and Strong Laws of Large Numbers in the setting of separable Banach spaces. If the random elements take values in a complete separable subspace of $D[0,1]$, for example in $C[0,1]$, in the norm topology, then uniform tightness of the sequence is equivalent to uniform convex tightness.

2. CONVERGENCE IN $D[0,1]$

The main purpose of this section is to study relationship between pointwise convergence, convergence in the Skorokhod topology and norm convergence of sequences in $D[0,1]$. This study is helpful in establishing some Weak Law of Large Numbers in $D[0,1]$ using pointwise Weak Law of Large Numbers for sequences of random elements in $D[0,1]$. The results established in this section on the relationship generalize certain results in the literature proved in this direction. These improvements will be pointed out as and when the occasion arises.

Recall that K is the collection of all relatively compact subsets of $D[0,1]$ whose convex hulls are also relatively compact in the Skorokhod topology. The following definitions are helpful to gain a good understanding of sets in K .

(2.1) For any $H \subset D[0,1]$ and x in $D[0,1]$, define

$$\omega_x(H) = \sup_{s,t \in H} |x(s) - x(t)|.$$

(2.2) For any x in $D[0,1]$ and $0 \leq \delta \leq 1$, define

$$\omega_x(\delta) = \sup_{0 \leq t \leq 1-\delta} \omega_x([t, t+\delta]).$$

(Modulus of continuity of x .)

(2.3) For any x in $D[0,1]$ and $0 < \delta < 1$, define

$$\omega'_x(\delta) = \inf_{\{t_i\}} \max_{0 < i \leq N} \omega_x([t_{i-1}, t_i]).$$

where the infimum is taken over all partitions

$0 = t_0 < t_1 < \dots < t_N = 1$ of $[0,1]$ satisfying

$t_i - t_{i-1} > \delta$ for all $i = 1, 2, \dots, N$.

(2.4) For any partition $0 = t_0 < t_1 < \dots < t_N = 1$ of

$[0,1]$, let $\langle t_{i-1}, t_i \rangle = [t_{i-1}, t_i]$ for $i = 1, 2, \dots, N-1$

and $\langle t_{N-1}, t_N \rangle = [t_{N-1}, t_N]$.

(2.5) For any set $A \subset D[0,1]$ and $\epsilon > 0$, let

$$S_\epsilon(A) = \{t \in [0,1] ; \sup_{x \in A} |x(t) - x(t-0)| > \epsilon\}.$$

$S_\epsilon(A)$ describes jumps of functions in A .

The following result characterizes sets in K .

Theorem 2.1 The following statements are equivalent.

(i) $K \in K$.

(ii) $S_\epsilon(K)$ is finite for every $\epsilon > 0$.

(iii) For every $\epsilon > 0$, there exists a partition $0 = t_0$
 $t_1 < \dots < t_N = 1$ of $[0,1]$ such that

$$\max_{1 \leq i \leq N} \omega_x([t_{i-1}, t_i]) \leq \epsilon$$

for every x in K .

Proof. This result is known. Equivalence of (i) and (ii) is proved by Daffer and Taylor [4, Theorem 6, p.92]. Equivalence of (i) and (iii) is proved by Daffer [3, Theorem 3.6, p.508]. We give a simple proof of the equivalence of (i) and (iii) exploiting the compactness property of $[0,1]$. The implication (iii) \implies (i)

easily follows from the equivalence of (i) and (ii). We prove

(i) \implies (iii). Let $\epsilon > 0$. By the equivalence of (i) and (ii),

and Lemma 7 of Daffer and Taylor [4, p.92], for every t in $[0,1]$,

there exist t' and t'' in $[0,1]$ such that $t'' < t < t'$,

$\sup_{x \in K} \omega_x([t, t']) \leq \epsilon$ and $\sup_{x \in K} \omega_x((t'', t)) \leq \epsilon$. For $t = 1$,

we note that $\sup_{x \in K} \omega_x((1'', 1]) \leq \epsilon$. Thus we have $\{[0, 0'), (1'', 1]\} \cup \{(t'', t') ; t \in (0, 1)\}$ as an open cover for $[0,1]$. There exists

a finite sub-cover $\{[0, 0'), (1'', 1]\} \cup \{(r_i'', r_i') ; r_i \in (0, 1)\}$,

$i = 1, 2, \dots, m\}$ for $[0,1]$. Let $t_0 = 0$ and $t_1 = 0'$. Then we

must have some r_i such that $r_i'' < 0' < r_i'$. If $r_i \leq 0'$, let $t_2 = r_i'$. Otherwise, let $t_2 = r_i$ and $t_3 = r_i'$. Continuing this way, we obtain a partition $0 = t_0 < t_1 < \dots < t_N = 1$ of $[0,1]$ such that $\max_{1 \leq i \leq N} \omega_x([t_{i-1}, t_i]) \leq \epsilon$ for every x in K . This completes the proof.

It is well known that if a sequence x_n , $n \geq 1$ in $D[0,1]$ converges to an element x_0 in $D[0,1]$ in the Skorokhod topology, then $x_n(t)$, $n \geq 1$ converges to $x_0(t)$ for every t which is a continuity point of x_0 . If x_0 is continuous, then convergence in the Skorokhod topology implies convergence in the norm topology. See Billingsley [1, p.112]. It is useful to find some conditions under which pointwise convergence, convergence in the Skorokhod topology and convergence in the norm topology are equivalent. The following results attend to this problem.

Theorem 2.2 Let a subset K of $D[0,1]$ have the following property.

$$\lim_{\delta \rightarrow 0} \sup_{x \in K} \omega'_x(\delta) = 0.$$

Let x_n , $n \geq 1$ be a sequence in K and x_0 in $D[0,1]$ continuous.

Then the following statements are equivalent.

(i) $\lim_{n \rightarrow \infty} x_n(t) = x_0(t)$ for every t in S for some dense

subset S of $[0,1]$.

(ii) $\lim_{n \rightarrow \infty} d(x_n, x_0) = 0$.

(iii) $\lim_{n \rightarrow \infty} ||x_n - x_0|| = 0$.

Proof. We need to prove only (i) \implies (iii). By the given

hypothesis, it is obvious that $\lim_{\delta \rightarrow 0} \sup_{x \in K} \omega_x([0, \delta]) = 0$ and

$\lim_{\delta \rightarrow 0} \sup_{x \in K} \omega_x([1-\delta, 1]) = 0$. By (i) and since x_0 is continuous,

it follows that

$$\lim_{n \rightarrow \infty} x_n(0) = x_0(0) \text{ and } \lim_{n \rightarrow \infty} x_n(1) = x_0(1). \quad (2.6)$$

Let $\epsilon > 0$. By the given hypothesis, there exists $\delta > 0$ such that

$$\omega_x'(\delta) < \epsilon \text{ for every } x \text{ in } K. \quad (2.7)$$

Since x_0 is uniformly continuous, there exists a partition $0 = t_0$

$< t_1 < \dots < t_m = 1$ of $[0,1]$ such that $|t_i - t_{i-1}| \leq \delta$ for

$i = 1, 2, \dots, m$, t_1, t_2, \dots, t_{m-1} in S and

$$\omega_{x_0}([t_{i-1}, t_i]) < \epsilon \quad (2.8)$$

for every $i = 1, 2, \dots, m$. By (i) and (2.6), we can find $N \geq 1$

such that

$$|x_n(t_i) - x_0(t_i)| < \epsilon \quad (2.9)$$

for every $i = 0, 1, 2, \dots, m$ whenever $n \geq N$. Choose and fix $n \geq N$

which is otherwise arbitrary. We claim that for every $i = 0, 1, \dots, m-1$, there exists u_i in $[t_i, t_{i+1}]$ such that

$$\omega_{x_n}([t_i, u_i]) \leq \epsilon \text{ and } \omega_{x_n}([u_i, t_{i+1}]) \leq \epsilon. \quad (2.10)$$

By (2.7), $\omega_{x_n}^*(\delta) < \epsilon$. There exists a partition $0 = v_0 < v_1 < \dots$

$< v_p = 1$ of $[0, 1]$ such that $v_i - v_{i-1} > \delta$ for every $i = 1, 2, \dots$

\dots, m and $\omega_{x_n}([v_{i-1}, v_i]) < \epsilon$ for every $i = 1, 2, \dots, m$. First,

look at the interval $[t_0, t_1]$. Since $t_1 - t_0 \leq \delta$ and $v_1 - v_0 > \delta$,

we have $[t_0, t_1] \subset [v_0, v_1]$. Take $u_0 = t_1$. Thus (2.10) is

satisfied for $i = 0$. Next, look at the interval $[t_1, t_2]$. If

$[t_1, t_2] \subset [v_0, v_1]$, take $u_1 = t_2$. If $[t_1, t_2]$ is not contained in

$[v_0, v_1]$, then $v_1 \leq t_2 < v_2$. In this case, take $u_1 = v_1$ and $u_2 = v_2$.

In any case, we observe that (2.10) is satisfied for $i = 1$.

Continuing this way, we see that the claim is justified. By (2.8),

(2.9) and (2.10), we obtain

$$\omega_{x_n - x_0}([t_i, u_i]) < 2\epsilon \text{ and } \omega_{x_n - x_0}([u_i, t_{i+1}]) < 2\epsilon$$

for $i = 0, 1, 2, \dots, m-1$, and

$$|x_n(t) - x_0(t)| < 4\epsilon \quad \text{whenever } t_i \leq t \leq t_{i+1}$$

for $i = 0, 1, 2, \dots, m-1$. This implies that $\|x_n - x_0\| < 4\epsilon$.

This last inequality is valid whenever $n \geq N$. Thus we have

$$\lim_{n \rightarrow \infty} \|x_n - x_0\| = 0 \quad \text{as desired.}$$

Remarks. The condition on K in the above theorem figures in a characterization of relatively compact subsets of $D[0,1]$ in the Skorokhod topology. More precisely, a subset K of $D[0,1]$ is relatively compact if and only if $\lim_{\delta \rightarrow 0} \sup_{x \in K} \omega'_x(\delta) = 0$ and $\sup_{x \in K} \|x\| < \infty$. See Billingsley [1, Theorem 14.3, p.116].

Using similar ideas as in the proof of the above theorem, one can prove the following theorem.

Theorem 2.3 Let K be a subset of $D[0,1]$ have the following property.

$$\lim_{\delta \rightarrow 0} \sup_{x \in K} \omega'_x(\delta) = 0.$$

Let x_n , $n \geq 1$ be a sequence in K and $x_0 \in D[0,1]$. Then

$\lim_{n \rightarrow \infty} \|x_n - x_0\| = 0$ if and only if $\lim_{n \rightarrow \infty} x_n(t) = x_0(t)$ for every t in $[0,1]$.

Remarks. Daffer [3, Theorem 3.1, p.504] showed the equivalence of convergence in the Skorokhod topology and convergence in the norm topology for sequences in a compact convex subset of $D[0,1]$.

Theorem 2.3 above shows the equivalence of norm convergence and pointwise convergence under conditions much less restrictive than those imposed by Daffer. However, the following theorem generalizes Theorem 3.1 of Daffer and the proof offered is much simpler than that offered by Daffer.

Theorem 2.4 Let $K \in \mathcal{K}$. Let $x_n, n \geq 1$ be a sequence in K .

Then the following statements are equivalent.

(i) $\lim_n x_n(t) = x_0(t)$ for every t in S for some dense subset S of $[0,1]$ for some x_0 in $D[0,1]$.

(ii) $\lim_{n \rightarrow \infty} d(x_n, x_0) = 0$.

(iii) $\lim_{n \rightarrow \infty} \|x_n - x_0\| = 0$.

Proof. The implications (iii) \implies (ii) \implies (i) are trivial.

We prove (i) \implies (iii). For any $\epsilon > 0$, by Theorem 2.1, there

exists a partition $0 = t_0 < t_1 < \dots < t_N = 1$ of $[0,1]$ such that

$$\max_{1 \leq i \leq N} \omega_x(t_{i-1}, t_i) < \epsilon \text{ for every } x \text{ in } K$$

and a partition $0 = s_0 < s_1 < \dots < s_N = 1$ of $[0,1]$ such that

$$\max_{1 \leq i \leq N} \omega_{x_0}(\langle s_{i-1}, s_i \rangle) < \epsilon ,$$

following the notation of (2.4). Putting these two partitions

together, we have the partition $0 = u_0 < u_1 < \dots < u_N = 1$ of $[0,1]$ such that

$$\sup_{x \in K \{x_0\}} \max_{1 \leq i \leq N} \omega_x(\langle u_{i-1}, u_i \rangle) < \epsilon .$$

For each $i = 1, 2, \dots, N$, choose r_i in S such that $u_{i-1} < r_i < u_i$ and then choose $M \geq 1$ such that whenever $n \geq M$, we have

$$\max_{0 \leq i \leq N} |x_n(r_i) - x_0(r_i)| < \epsilon .$$

Thus, when $n \geq M$, we have

$$\begin{aligned} \|x_n - x_0\| &= \max_{1 \leq i \leq N} \sup_{t \in \langle u_{i-1}, u_i \rangle} |x_n(t) - x_0(t)| \\ &= \max_{1 \leq i \leq N} \sup_{t \in \langle u_{i-1}, u_i \rangle} |x_n(t) - x_n(r_i) \\ &\quad + x_n(r_i) - x_0(r_i) + x_0(r_i) - x_0(t)| \\ &< 3\epsilon . \end{aligned}$$

This completes the proof.

3. ON STOCHASTIC CONVERGENCE

Let x_n , $n \geq 1$ be a sequence of random elements defined on a probability space (Ω, \mathcal{B}, P) taking values in $D[0,1]$. Let a_{nk} , $n \geq 1$, $1 \leq k \leq n$ be a triangular array of real numbers. In this section, we are interested in studying the convergence of the sequence $\sum_{k=1}^n a_{nk} x_k$, $n \geq 1$ of weighted sums in some sense (either in probability, or in p -th mean or almost surely $[P]$) either in the Skorokhod metric d or in the uniform norm $\|\cdot\|$. More specifically, we ask whether the convergence of

$$\sum_{k=1}^n a_{nk} x_k(\cdot)(t), n \geq 1 \text{ to } 0$$

for every t in some dense subset S of $[0,1]$ in some sense would force the convergence of

$$\sum_{k=1}^n a_{nk} x_k, n \geq 1 \text{ to } 0$$

in the same sense. We answer this question in this section. (Note

that $\sum_{k=1}^n a_{nk} x_k(\cdot)(t), n \geq 1$ is a sequence of real random variables for every t in $[0,1]$.) Had $D[0,1]$ been separable in the norm topology $\|\cdot\|$, or Frechet space in the Skorokhod

topology, many of the classical techniques would have become applicable to derive some Weak and Strong Laws of Large Numbers in $D[0,1]$. In separable Banach spaces, for the validity of certain Weak Law of Large Numbers analogous to those available on the real line, a crucial condition imposed in the literature is "uniformly compactly r -th order integrability" of the sequence of random elements under discussion. A sequence $x_n, n \geq 1$ of random elements taking values in a separable Banach space B is said to be uniformly compactly r -th order integrable ($r \geq 0$) if for every $\epsilon > 0$ there exists a compact subset C of B such that

$$\int_{\{x_n \in C\}} ||x_n||^r dP < \epsilon$$

for every $n \geq 1$. See Wang and Bhaskara Rao [12] for some of the ramifications of this definition and the attendant limit theorems. See also Hoffmann-Jørgensen and Pisier [5, Theorem 2.4, p.592]. In Frechet or Banach spaces, closure of the convex hull of a compact set is compact. But this is not true in the Skorokhod

topology of $D[0,1]$. See Daffer and Taylor [4, p.91]. In order to work out a Weak Law of Large Numbers in $D[0,1]$, a condition similar to "uniformly compactly r -th order integrability" is needed to be imposed on the sequence $x_n, n \geq 1$ in $D[0,1]$. Taylor and Daffer [10, p.99] introduced the following "(CT) condition".

A sequence $x_n, n \geq 1$ of random elements in $D[0,1]$ is said to satisfy (CT) condition if for every $\epsilon > 0$ there exists a compact convex subset K of $D[0,1]$ satisfying

$$\int_{\{x_n \in K^c\}} ||x_n|| dP < \epsilon$$

for every $n \geq 1$. Taylor and Daffer [10, p.99] observed that if $x_n, n \geq 1$ is uniformly convex tight, i.e., for every $\epsilon > 0$ there exists a compact convex subset K of $D[0,1]$ such that $P\{x_n \in K^c\} < \epsilon$ for every $n \geq 1$ and $\sup_{n \geq 1} E||x_n||^p < \infty$ for some $p > 1$, then $x_n, n \geq 1$ satisfies (CT) condition. One of the goals of this section is to characterize precisely this (CT) condition.

The following theorem implies that $x_n, n \geq 1$ satisfies (CT) condition if and only if $x_n, n \geq 1$ is uniformly convex tight

and $\{||x_n||, n \geq 1\}$ is uniformly integrable.

Theorem 3.1 Let $x_n, n \geq 1$ be a sequence of $D[0,1]$ -valued random elements and $r > 0$. Then the following two statements (i) (ii) are equivalent.

(i) For every $\epsilon > 0$, there exists a Borel set K in K such that

$$\int_{K^c} ||x_n||^r dP < \epsilon$$

for every $n \geq 1$.

(ii)(a) For every $\epsilon > 0$, there exists a Borel set K in K such that $P\{x_n \in K\} \geq 1 - \epsilon$ for every $n \geq 1$, i.e., $x_n, n \geq 1$ is uniformly convex tight.

(b) $\{||x_n||^r, n \geq 1\}$ is uniformly integrable.

Proof. Equivalence of (i) and (ii) is obvious when $r = 0$. Let $r > 0$. It is not hard to see that (ii) \implies (i) and (i) \implies (ii)(b). We prove (i) \implies (ii)(a). By (i), for every $\epsilon > 0$ and $i = 1, 2, 3, \dots$, there exists a Borel set K_i^c in K such that

$$\int_{(K_i^c)^c} ||x_n||^r dP < \epsilon/2^i \cdot i^r$$

for every $n \geq 1$. Let $K_i = \bigcup_{j=1}^i K_j^c$, $i \geq 1$. Then $K_i \in K$. See the remarks following Theorem 2.6 of [3]. Let $S_0 = D[0,1]$ and $S_i = \{x \in D[0,1] ; ||x|| \leq 1/i\}$ for $i = 1, 2, 3, \dots$. Then

$$P\{x_n \in (K_i \cup S_i)^c\} \leq i^r \int_{(K_i \cup S_i)^c} ||x_n||^r dP < i^r (\epsilon/2^i \cdot i^r) = \epsilon/2^i$$

for every $n \geq 1$ and $i \geq 1$. This implies that, for every $n \geq 1$,

$$\begin{aligned} P\{x_n \in \bigcap_{i \geq 1} (K_i \cup S_i)^c\} &= 1 - P\{x_n \in \bigcup_{i \geq 1} (K_i \cup S_i)^c\} \\ &\geq 1 - \sum_{i \geq 1} P\{x_n \in (K_i \cup S_i)^c\} \\ &\geq 1 - \sum_{i \geq 1} \epsilon/2^i = 1 - \epsilon. \end{aligned}$$

We show that $B = \bigcap_{i \geq 1} (K_i \cup S_i) \in K$. Now, $\overline{B} = \overline{\bigcap_{i \geq 1} (K_i \cup S_i)} \subset$

$\bigcap_{i \geq 1} (\overline{K_i} \cup S_i)$, after observing that $S_i = \{x \in D[0,1] ; ||x|| \leq 1/i\}$,

$= \{x \in D[0,1] ; d(x, 0) \leq 1/i\}$ and consequently, that S_i is

closed in the Skorokhod topology. The set $\bigcap_{i \geq 1} (\overline{K_i} \cup S_i)$ is

compact since it is closed and totally bounded in the d -metric.

This implies that B is relatively compact. We show that for any

$\epsilon > 0$, $S_\epsilon(B)$ is finite. Observe that, since $S_0 \supset S_1 \supset S_2 \supset \dots$

and $K_1 \subset K_2 \subset K_3 \subset \dots$,

$$B = \bigcap_{i \geq 1} (K_i \cup S_i) = \{0\} \cup \left(\bigcup_{i \geq 0} (S_i \cap S_{i+1}^c \cap K_{i+1}) \right).$$

For any $\epsilon > 0$,

$$S_\epsilon(B) = \{t \in [0,1] : \sup_{x \in B} |x(t) - x(t-0)| > \epsilon\}$$

$$= S_\epsilon \left(\bigcup_{i=0}^{[2/\epsilon]} (S_i \cap S_{i+1}^c \cap K_{i+1}) \right)$$

$$\subset \bigcup_{i=0}^{[2/\epsilon]} S_\epsilon(K_{i+1}).$$

where $[2/\epsilon]$ is the largest integer $\leq 2/\epsilon$. Since $K_i \in K$ for $i = 1, 2, 3, \dots$, by Theorem 2.1, $S_\epsilon(K_i)$ is finite. Hence $S_\epsilon(B)$ is finite. By Theorem 2.1, $B \in K$. This completes the proof.

Now, we concentrate on proving some Laws of Large Numbers.

To do this, we resort to the classical truncation technique by truncating the random elements to a set K in K . The following result is useful in this connection.

Theorem 3.2 Let $X_n, n \geq 1$ be a sequence of $D[0,1]$ -valued random elements such that $P\{X_n \in K\} = 1$ for every $n \geq 1$, for some $\bigwedge K$ in K . Let $a_{nk}, 1 \leq k \leq n, n \geq 1$ be a triangular array of

real numbers such that $\sum_{k=1}^n |a_{nk}| \leq r$ for all $n \geq 1$ for some

positive constant r . Then

$\sum_{k=1}^n a_{nk} x_k, n \geq 1$ converges to 0 in probability

(in r -th mean)(a.e. [P])

if and only if

$\sum_{k=1}^n a_{nk} x_k(\cdot)(t), n \geq 1$ converges to 0 in probability

(in r -th mean)(a.e. [P])

for every t in S for some dense subset S of $[0,1]$.

Proof. We need to prove the "if part" only. For any $\epsilon > 0$, by

Theorem 2.1, there exists a partition $T = \{t_i\}_{i=0}^N$ of $[0,1]$

such that $\max_{1 \leq i \leq N} \omega_x([t_{i-1}, t_i]) \leq \epsilon/r$ for every x in K .

Choose s_1, s_2, \dots, s_N in S such that $t_{i-1} \leq s_i < t_i$ for

each $i = 1, 2, \dots, N$. Then $|x(t) - x(s_i)| \leq \epsilon/r$ whenever

$t \in (t_{i-1}, t_i)$ and $x \in K$. Let $Y_T : D[0,1] \rightarrow D[0,1]$ be defined by

$$Y_T(x)(t) = \sum_{i=1}^N x(s_i) I_{(t_{i-1}, t_i)}(t), \quad 0 \leq t \leq 1.$$

Then for every ω in Ω ,

$$\begin{aligned}
 \left| \left| \sum_{k=1}^n a_{nk} x_k(\omega) \right| \right| &\leq \left| \left| \sum_{k=1}^n a_{nk} (x_k(\omega) - Y_T(x_k(\omega))) \right| \right| \\
 &\quad + \left| \left| \sum_{k=1}^n a_{nk} Y_T(x_k(\omega)) \right| \right| \\
 &\leq \epsilon + \max_{1 \leq i \leq N} \left| \sum_{k=1}^n a_{nk} x_k(\omega)(s_i) \right|.
 \end{aligned}$$

In the above chain of inequalities, we have used the information that $P\{X_n \in K\} = 1$ for every $n \geq 1$ and the fact that $\|x - Y_T(x)\| \leq \epsilon/\Gamma$ for every x in K . The desired conclusion now easily follows from the above chain of inequalities.

Corollary 3.3 (Taylor and Daffer [10, Theorem 3, p.102]) Let K be a compact convex subset of $D[0,1]$. Let x_n , $n \geq 1$ be a ~~independent~~ sequence of random elements with $EX_n = 0$ and $P\{X_n \in K\} = 1$ for every $n \geq 1$. Let a_{nk} , $1 \leq k \leq n$, $n \geq 1$ be a triangular array of real numbers satisfying $\sum_{k=1}^n |a_{nk}| \leq \Gamma$ for every $n \geq 1$ for some positive constant Γ and $\max_{1 \leq k \leq n} |a_{nk}| = O(n^{-\alpha})$ as $n \rightarrow \infty$ for some $\alpha > 0$. Then

$$\left| \left| \sum_{k=1}^n a_{nk} x_k \right| \right|, \quad n \geq 1 \text{ converges to } 0 \text{ a.e. } [P].$$

Proof. By Theorem 2 of Rohatgi [7, p.306], the sequence of random variables $\sum_{k=1}^n a_{nk} x_k(\cdot)(t)$, $n \geq 1$ converges to 0 a.e. $[P]$,

for every t in $[0,1]$. Now, an application of Theorem 3.2 completes the proof.

Now we prove a general result from which some Weak Laws of Large Numbers can be derived.

Theorem 3.4 Let $x_n, n \geq 1$ be a sequence of random elements taking values in $D[0,1]$ satisfying either (i) or (ii) of Theorem 3.1 for $r = 1$. Let $a_{nk}, 1 \leq k \leq n, n \geq 1$ be a triangular array of real numbers satisfying $\sum_{k=1}^n |a_{nk}| \leq r$ for every $n \geq 1$ for some positive constant r . In the following statements, then (i) \Rightarrow (iv) \Rightarrow (iii) \Rightarrow (ii).

(i) $\sum_{k=1}^n a_{nk}(x_k(\cdot)(t) - EX_k(\cdot)(t)), n \geq 1$ converges to 0 in probability for every t in S for some dense subset of $[0,1]$.

(ii) $d\left(\sum_{k=1}^n a_{nk}x_k, \sum_{k=1}^n a_{nk}EX_k\right), n \geq 1$ converges to 0 in probability.

(iii) $E d\left(\sum_{k=1}^n a_{nk}x_k, \sum_{k=1}^n a_{nk}EX_k\right), n \geq 1$ converges to 0.

(iv) $E \left| \left| \sum_{k=1}^n a_{nk}(x_k - EX_k) \right| \right|, n \geq 1$ converges to 0.

Proof. First, we prove (i) \implies (iv). Let $\epsilon > 0$. By (i) of

Theorem 3.1, choose K in K such that

$$E \left| \left| x_n I_{\{x_n \in K^c\}} \right| \right| < \epsilon / 12r$$

for every $n \geq 1$. As in the proof of Theorem 3.2, we can choose

s_1, s_2, \dots, s_N in S and the operator Y_T built on s_1, s_2, \dots, s_N such that

$$\sup_{x \in K} \left| \left| x - Y_T(x) \right| \right| < \epsilon / 6r .$$

We note that

$$\begin{aligned} E \left| \left| \sum_{k=1}^n a_{nk} (x_k - E x_k) \right| \right| &\leq E \left| \left| \sum_{k=1}^n a_{nk} (x_k - Y_T(x_k)) \right| \right| \\ &\quad + E \left| \left| \sum_{k=1}^n a_{nk} (Y_T(x_k) - Y_T(E x_k)) \right| \right| \\ &\quad + E \left| \left| \sum_{k=1}^n a_{nk} (Y_T(E x_k) - E x_k) \right| \right| \quad (3.1) \end{aligned}$$

We show that each of the expressions on the right hand side of

the above inequality can be made $< \epsilon / 3$ for all large n . Consider

$$\begin{aligned} E \left| \left| \sum_{k=1}^n a_{nk} (x_k - Y_T(x_k)) \right| \right| &\leq E \left| \left| \sum_{k=1}^n a_{nk} (x_k I_{\{x_k \in K\}} - Y_T(x_k I_{\{x_k \in K\}})) \right| \right| \\ &\quad + E \left| \left| \sum_{k=1}^n a_{nk} x_k I_{\{x_k \in K^c\}} \right| \right| \\ &\quad + E \left| \left| \sum_{k=1}^n a_{nk} Y_T(x_k I_{\{x_k \in K^c\}}) \right| \right| \end{aligned}$$

- 28 -

$$\begin{aligned} & \leq \sum_{k=1}^n |a_{nk}| \sup_{x \in K} ||x - Y_T(x)|| \\ & + 2 \sum_{k=1}^n |a_{nk}| E ||x_k I_{\{x_k \in K^c\}}|| \\ & < \epsilon/6 + \epsilon/6 = \epsilon/3 \end{aligned} \quad (3.2)$$

In the above chain of inequalities, we have used the fact that

$||Y_T(x)|| \leq ||x||$ for every x in $D[0,1]$. Now, consider

$$\begin{aligned} ||\sum_{k=1}^n a_{nk}(Y_T(Ex_k) - Ex_k)|| & \leq ||\sum_{k=1}^n a_{nk}(Y_T(Ex_k I_{\{x_k \in K\}}) - Ex_k I_{\{x_k \in K\}})|| \\ & + ||\sum_{k=1}^n a_{nk}(Y_T(Ex_k I_{\{x_k \in K^c\}}) - \\ & \quad Ex_k I_{\{x_k \in K^c\}})|| \\ & \leq \sum_{k=1}^n |a_{nk}| ||Y_T(Ex_k I_{\{x_k \in K\}}) - Ex_k I_{\{x_k \in K\}}|| \\ & + 2 \sum_{k=1}^n |a_{nk}| E ||x_k I_{\{x_k \in K^c\}}|| \\ & < \sum_{k=1}^n |a_{nk}| E ||Y_T(x_k I_{\{x_k \in K\}}) - x_k I_{\{x_k \in K\}}|| \\ & + \epsilon/6 \\ & \leq \sum_{k=1}^n |a_{nk}| \sup_{x \in K} ||Y_T(x) - x|| + \epsilon/6 \\ & < \epsilon/6 + \epsilon/6 = \epsilon/3. \end{aligned} \quad (3.3)$$

In the chain of inequalities above, we have used the fact that

$Y_T(Ex) = EY_T(x)$ for any random element with $E||x|| < \infty$. Next,

we use the hypothesis on the pointwise convergence in probability.

For any t in $[0,1]$ and $n \geq 1$,

$$\left| \sum_{k=1}^n a_{nk}(x_k(\cdot)(t) - EX_k(\cdot)(t)) \right| \leq \sum_{k=1}^n |a_{nk}| (||x_k|| + E||x_k||).$$

This implies that the sequence $\sum_{k=1}^n a_{nk}(x_k(\cdot)(t) - EX_k(\cdot)(t))$, $n \geq 1$

is uniformly integrable for every t in $[0,1]$. Hence it follows that this sequence converges in mean to 0, i.e.,

$$E \left| \sum_{k=1}^n a_{nk}(x_k(\cdot)(t) - EX_k(\cdot)(t)) \right|, \quad n \geq 1 \quad \text{converges to 0}$$

for every t in S . We can find $N_0 \geq 1$ such that

$$E \left| \sum_{k=1}^n a_{nk}(x_k(\cdot)(s_i) - EX_k(\cdot)(s_i)) \right| < \epsilon/3N$$

for all $i = 1, 2, \dots, N$ whenever $n \geq N_0$. Consequently, we have

$$\begin{aligned} E \left| \left| \sum_{k=1}^n a_{nk}(Y_T(x_k) - Y_T(EX_k)) \right| \right| \\ \leq \sum_{i=1}^N E \left| \sum_{k=1}^n a_{nk}(x_k(\cdot)(s_i) - EX_k(\cdot)(s_i)) \right| < \epsilon/3 \end{aligned} \quad (3.4)$$

whenever $n \geq N_0$. Finally, (3.1), (3.2), (3.3) and (3.4) yield

$$E \left| \left| \sum_{k=1}^n a_{nk}(x_k - EX_k) \right| \right| < \epsilon \quad \text{whenever } n \geq N_0.$$

This completes the proof. (iv) \implies (iii) follows from $d(x, y) \leq ||x - y||$ for all x, y in $D[0,1]$.

Remarks. (1) Taylor and Daffer [10, Theorem 1, p.97] proved the implication (i) \implies (ii) under the additional assumption that

the set S is the set of all dyadic rationals in $[0,1]$.

(2) As has been remarked by Taylor and Daffer [10, p.99],

(ii) \implies (i) if the sequence $\sum_{k=1}^n a_{nk} E X_k$, $n \geq 1$ converges in

Skorokhod topology. In that case, we have (i), (ii), (iii) and (iv) of Theorem 3.4 above are all equivalent.

(3) Theorem 3, Corollary 4 and Theorem 5 of Daffer and Taylor [4, p. 90-91] are special cases of the above result.

See also Theorem 1, Theorem 2 and Corollary of Taylor and Daffer [9, p.412-415].

(4) In the discussion following Theorem 2 of Taylor and Daffer [9, p.415], it is argued that

$$\left\| \left(\frac{1}{n} \right) \sum_{k=1}^n X_k - E X_1 \right\|, n \geq 1 \text{ converges to } 0$$

in probability under the additional assumption that $E X_1$ is continuous. According to Theorem 3.4, this assumption is not necessary. We have the stronger conclusion that $E \left\| \left(\frac{1}{n} \right) \sum_{k=1}^n X_k - E X_1 \right\|$, $n \geq 1$ converges to 0.

As has been mentioned in the introductory remarks for Theorem 3.4, Theorem 3.4 is useful in deriving some Weak Law of

Large Numbers. We give a sample Weak Law of Large Numbers below.

Theorem 3.5 Let $x_n, n \geq 1$ be a sequence of pairwise independent random elements in $D[0,1]$ satisfying either (i) or (ii) of for $r = 1$.

Theorem 3.1A Let $a_{nk}, 1 \leq k \leq n, n \geq 1$ be a triangular array of real numbers satisfying $\sum_{k=1}^n |a_{nk}| \leq r$ for every $n \geq 1$ for some positive constant r and $\max_{1 \leq k \leq n} |a_{nk}|, n \geq 1$ converges to 0. Then

$$\lim_{n \rightarrow \infty} E \left| \left| \sum_{k=1}^n a_{nk} (x_k - E x_k) \right| \right| = 0.$$

Proof. By Theorem 3.4 above, it is enough to show that for any t in $[0,1]$, $\sum_{k=1}^n a_{nk} (x_k(\cdot)(t) - E x_k(\cdot)(t)), n \geq 1$ converges to 0 in probability. For a given $\epsilon > 0$ and $\delta > 0$, choose a Borel set K in such that

$$E \left| \left| x_n I_{\{x_n \in K^c\}} \right| \right| < \epsilon \delta / 8r$$

for all $n \geq 1$. Choose $N \geq 1$ such that

$$\max_{1 \leq k \leq n} |a_{nk}| < \delta \epsilon^2 / 8 \theta^2 r$$

for every $n \geq N$, where $\theta = \sup_{x \in K} |x|$. We observe that if $n \geq N$

$$P \left\{ \left| \sum_{k=1}^n a_{nk} (x_k(\cdot)(t) - E x_k(\cdot)(t)) \right| \right\} \leq$$

$$\begin{aligned}
 & P\left\{ \left| \sum_{k=1}^n a_{nk} (x_k(\cdot)(t) I_{\{x_k \in K\}}(\cdot)(t) - E x_k(\cdot)(t) I_{\{x_k \in K\}}(\cdot)(t)) \right| > \epsilon/2 \right\} \\
 & + P\left\{ \left| \sum_{k=1}^n a_{nk} (x_k(\cdot)(t) I_{\{x_k \in K^c\}}(\cdot)(t) - E x_k(\cdot)(t) I_{\{x_k \in K^c\}}(\cdot)(t)) \right| > \epsilon/2 \right\} \\
 & \leq (4/\epsilon^2) \sum_{k=1}^n |a_{nk}|^2 \text{Var}(x_k(\cdot)(t) I_{\{x_k \in K\}}(\cdot)(t)) \\
 & + (4/\epsilon) \sum_{k=1}^n |a_{nk}| E|x_k(\cdot)(t) I_{\{x_k \in K^c\}}(\cdot)(t)| \\
 & \leq (4/\epsilon^2) \max_{1 \leq k \leq n} |a_{nk}| \sum_{k=1}^n |a_{nk}| (\sup_{x \in K} ||x||)^2 \\
 & + (4/\epsilon) \sum_{k=1}^n |a_{nk}| E||x_k I_{\{x_k \in K^c\}}|| \\
 & < (4/\epsilon^2)(\delta \epsilon^2 / 8 \theta^2 \Gamma) \Gamma \theta^2 + (4/\epsilon) \Gamma (\delta \epsilon / 8 \Gamma) = \delta/2 + \delta/2 = \delta.
 \end{aligned}$$

This completes the proof.

Remark Taylor and Daffer [10, Theorem 2, p.100] established the conclusion of the above theorem under the much stronger condition that $\max_{1 \leq k \leq n} |a_{nk}| = O(n^{-\alpha})$ as $n \rightarrow \infty$ for some $\alpha > 0$ and $E x_k = 0$ for every $n \geq 1$. This Theorem 2 of Taylor and Daffer is thus a special case of the above theorem.

Now, we establish some Strong Law of Large Numbers. Our main goal is to seek analogues of Marcinkiewicz-Zygmund-Kolmogorov's and Brunk-Chung's Strong Laws of Large Numbers for $D[0,1]$ -valued

random elements. (See Chung [2, p.125] for a proof of Marcinkiewicz-Zygmund-Kolmogorov's Strong Law of Large Numbers. For a proof of Brunk-Chung's Strong Law of Large Numbers, see Chung [2, p. 348].) The following result gives the desired analogues. This result generalizes Theorem 1 of Daffer and Taylor [4, p.88].

Theorem 3.6 Let $x_n, n \geq 1$ be a sequence of independent $D[0,1]$ - valued random elements satisfying either (i) or (ii) of Theorem 3.1 for $r = 1$.

(a) If $1 \leq p \leq 2$ and $\sum_{j \geq 1} E||x_j||^p / j^p < \infty$, then

$||(1/n) \sum_{k=1}^n (x_k - EX_k)||, n \geq 1$ converges to 0 a.e. [P].

(b) If $p \geq 2$ and $\sum_{j \geq 1} E||x_j||^p / j^{1+p/2} < \infty$, then

$||(1/n) \sum_{k=1}^n (x_k - EX_k)||, n \geq 1$ converges to 0 a.e. [P].

Proof. (a) For a given $\epsilon > 0$, choose a Borel set K in $D[0,1]$ such that $E||x_n I_{\{x_n \in K^c\}}|| < \epsilon/4$ for every $n \geq 1$. As in the proof of Theorem 3.2, we can choose s_1, s_2, \dots, s_N in $[0,1]$ and build a linear operator $Y_T : D[0,1] \rightarrow D[0,1]$ such that

$$\sup_{x \in K} ||x - Y_T(x)|| < \epsilon/4 \quad (3.5)$$

Then

$$\begin{aligned}
 \left\| \left(\frac{1}{n} \sum_{k=1}^n (x_k - EX_k) \right) \right\| &\leq \left\| \left(\frac{1}{n} \sum_{k=1}^n (x_k I_{\{x_k \in K\}} - EX_k I_{\{x_k \in K\}}) \right) \right\| \\
 &\quad + \left\| \left(\frac{1}{n} \sum_{k=1}^n x_k I_{\{x_k \in K^c\}} \right) \right\| \\
 &\quad + \left\| \left(\frac{1}{n} \sum_{k=1}^n EX_k I_{\{x_k \in K^c\}} \right) \right\| \\
 &\leq \left\| \left(\frac{1}{n} \sum_{k=1}^n (x_k I_{\{x_k \in K\}} - EX_k I_{\{x_k \in K\}}) \right) \right\| \\
 &\quad + \left(\frac{1}{n} \sum_{k=1}^n (\left\| x_k I_{\{x_k \in K^c\}} \right\| - E \left\| x_k I_{\{x_k \in K^c\}} \right\|) \right) \\
 &\quad + \left(\frac{2}{n} \sum_{k=1}^n E \left\| x_k I_{\{x_k \in K^c\}} \right\| \right) \quad (3.6)
 \end{aligned}$$

We show that the first term on the left hand side of the above inequality is $< \epsilon/2$ as $n \rightarrow \infty$ a.e. [P], the second term converges to 0 a.e. [P] and the third term is, obviously $< \epsilon/2$. Using (3.5), we obtain

$$\begin{aligned}
 &\left\| \left(\frac{1}{n} \sum_{k=1}^n (x_k I_{\{x_k \in K\}} - EX_k I_{\{x_k \in K\}}) \right) \right\| \\
 &\leq \left\| \left(\frac{1}{n} \sum_{k=1}^n (x_k I_{\{x_k \in K\}} - Y_T(x_k I_{\{x_k \in K\}})) \right) \right\| \\
 &\quad + \left\| \left(\frac{1}{n} \sum_{k=1}^n (EX_k I_{\{x_k \in K\}} - Y_T(EX_k I_{\{x_k \in K\}})) \right) \right\| \\
 &\quad + \left\| \left(\frac{1}{n} \sum_{k=1}^n (Y_T(x_k I_{\{x_k \in K\}}) - Y_T(EX_k I_{\{x_k \in K\}})) \right) \right\| \\
 &\leq \epsilon/2 + \left\| \left(\frac{1}{n} \sum_{k=1}^n (Y_T(x_k I_{\{x_k \in K\}}) - Y_T(EX_k I_{\{x_k \in K\}})) \right) \right\|
 \end{aligned}$$

The last term on the left hand side of the above inequality is equal to

$$\begin{aligned} & \sum_{i=1}^N \left| \left(\frac{1}{n} \sum_{k=1}^n (x_k(\cdot)(s_i) I_{\{x_k \in K\}}(\cdot)(s_i) \right. \right. \\ & \quad \left. \left. - E x_k(\cdot)(s_i) I_{\{x_k \in K\}}(\cdot)(s_i) \right) \right| \end{aligned}$$

which, by Marcinkiewicz-Zygmund-Kolmogorov's Strong Law of Large Numbers, converges to 0 a.e. [P] as $n \rightarrow \infty$. Consequently,

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \left| \left| \left(\frac{1}{n} \sum_{k=1}^n (x_k I_{\{x_k \in K\}} - E x_k I_{\{x_k \in K\}}) \right) \right| \right| \\ & \leq \varepsilon/2 \quad \text{a.e. [P]} . \end{aligned} \quad (3.7)$$

Again, by Marcinkiewicz-Zygmund-Kolmogorov's Strong Law of Large Numbers, we have

$$\begin{aligned} & \left(\frac{1}{n} \sum_{k=1}^n \left(\left| x_k I_{\{x_k \in K\}} \right| - E \left| x_k I_{\{x_k \in K\}} \right| \right) \right), \quad n \geq 1 \\ & \text{converges to 0 a.e. [P]} . \end{aligned} \quad (3.8)$$

Thus, (3.6), (3.7) and (3.8) yield

$$\limsup_{n \rightarrow \infty} \left| \left| \left(\frac{1}{n} \sum_{k=1}^n (x_k - E x_k) \right) \right| \right| \leq \varepsilon \quad \text{a.e. [P]} .$$

Since ε is arbitrary, we have the desired result. Proof of (b) is similar to that of (a) and is omitted.

Remarks Daffer and Taylor [4, Theorem 1, p.88] established the conclusion of the above theorem under the assumptions that

for some $r > 1$.

x_n 's are independent, uniformly convex tight and $\sup_{n \geq 1} E||x_n||^r < \infty$.

In fact, they established the weaker conclusion that the almost sure convergence takes place in the Skorokhod topology. This result is a special case of the above theorem.

Using the argument presented in the proof of Theorem 3.6 above, one can obtain the $D[0,1]$ space version of Rohatgi's Theorem 2 [7, p.306] as reported below. This version generalizes Theorem 4 of Taylor and Daffer [10, p. 102].

Theorem 3.7 Let $x_n, n \geq 1$ be a sequence of $D[0,1]$ - valued random elements uniformly convex tight and uniformly dominated by a non-negative real random variable Y with $EY^r < \infty$ for some $r > 1$. Let $a_{nk}, 1 \leq k \leq n, n \geq 1$ be a triangular array of real numbers satisfying $\sum_{k=1}^n |a_{nk}| \leq r$ for every $n \geq 1$ for some positive constant r and $\max_{1 \leq k \leq n} |a_{nk}| = O(n^{-s})$ as $n \rightarrow \infty$ for some $0 < s < 1/r$.

(a) If $0 < s < 1$ and $x_n, n \geq 1$ is independent, then

$$\lim_{n \rightarrow \infty} \left| \left| \sum_{k=1}^n a_{nk} (x_k - E x_k) \right| \right| = 0 \text{ a.e. } [P].$$

(b) If $s \geq 1$ and $X_n, n \geq 1$ is pairwise independent, then

$$\lim_{n \rightarrow \infty} \left| \left| \sum_{k=1}^n a_{nk} (X_k - E X_k) \right| \right| = 0 \text{ a.e. [P].}$$

Remark For separable Banach space valued version of the above theorem, see Wang and Bhaskara Rao [12, Theorem 4.2].

Now, we establish an analogue of Jamison, Orey and Pruitt's Strong Law of Large Numbers in $D[0,1]$ space. See [6 , Theorem 3, p.42]. Let $a_n, n \geq 1$ be a sequence of positive numbers and $A_n = \sum_{i=1}^n a_i, n \geq 1$ with $\lim_{n \rightarrow \infty} \max_{1 \leq i \leq n} (a_i/A_n) = 0$. Let $N(n) = \text{card}\{i \geq 1; (A_i/a_i) \leq n\}, n \geq 1$.

Theorem 3.8 Let $X_n, n \geq 1$ be a sequence of pairwise independent identically distributed $D[0,1]$ -valued random elements. If

(a) $N(n)/n \leq r$ for every $n \geq 1$ for some positive constant r

and $E\|X_1\| < \infty$,

or

(b) $\sup_{n \geq 1} |a_n| < \infty$ and $E\|X_1\| \log^+ \|X_1\| < \infty$ holds, then

$$\lim_{n \rightarrow \infty} \left| \left| \sum_{i=1}^n (a_i/A_n) X_i - E X_1 \right| \right| = 0 \text{ a.e. [P].}$$

Proof. One can prove this result using the argument given in the proof of Theorem 7.3.2 of [8] and combining it with the argument given in the proof of Theorem 4 of Wang and Bhaskara Rao [1].

Remarks The above result generalizes Theorem 7.3.2 of Taylor [8].

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